Exam Lab-on-Chip, 6/06/2008.

1. Assume that the drag on a small sphere placed in a rapidly moving stream of fluid depends on the fluid density but not the fluid viscosity. Using dimensional analysis determine how the drag is affected if the velocity of the fluid is doubled.

Solution:

The problem variables are drag D, sphere diameter d, velocity v and fluid density ρ:

$$D = f\left(d, \rho, v\right)$$

We have 4 variables and 3 dimension, so we need just one Pi term and we can equate it to a constant: $\frac{D}{\rho d^2 v^2} = const$.

Answer: Drag will increase by a factor of 4.

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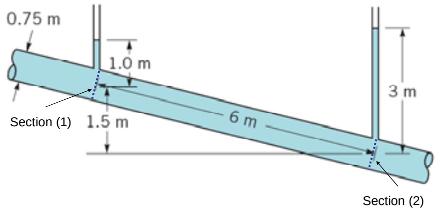
Water flows steadily through the inclined pipe as indicated in the Figure. Determine the following:
(a) the difference in pressure p_1-p_2;
(b) water flow direction and velocity;
(c) the axial and normal force exerted by the pipe wall on the flowing water between sections (1) and (2).



Solution:

(a). The pressure difference is as shown by the manometers on the figure:

$$\Delta p = p_2 - p_1 = \rho g (h_2 - h_1) = 19.6 \text{ kPa}$$

Note, that pressure at point 2 is higher.

(b). Assuming viscous laminar flow in the pipe we apply Poiseulle's law to obtain the average velocity from point 1 to point 2:

$$v = \frac{(\rho g \Delta h - \Delta p) \cdot d^2}{32 \mu l} = -1.4 \cdot 10^4 \ m/s$$

Note that the flow is in upward direction, from section 2 to section 1.

The velocity is very high so our laminar assumption is most probably not correct. To double check we can estimate Re number: $\mathrm{Re} = \frac{\rho v d}{\mu} \sim 10^{10}$. So, the flow is definitely turbulent and our initial approximation is wrong. At a such high Re numbers the flow will predominantly depend on the wall roughness; as we don't have any information on the roughness or the material of the tube we cannot go any further and have to stop at this point.

(c). Let's consider a control volume consisting of a tube between points 1 and 2. The force balance will include the pressures, the weight of the water and holding force F exerted by the pipe wall (note, that the same momentum is flowing in and out of the control volume, so no contribution from the momentum in this case):

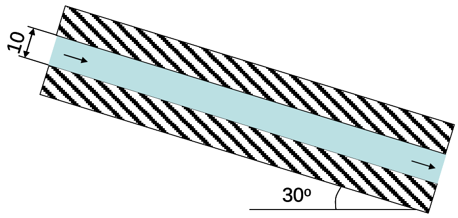
$$F_{axial} + \pi \frac{d^2}{4} (P_1 - P_2) + \rho g \pi \frac{d^2}{4} L_0 \cdot \sin(\alpha) = 0$$

$$F_{norm} - \rho g \pi \frac{d^2}{4} L_0 \cdot \cos(\alpha) = 0$$

Where the angle α can be obtained as: $\tan(\alpha) = 1.5/6$

Answer: $F_{axial} = 2.17\,\text{kN}$; $F_{norm} = 25.2\,\text{kN}$

2. Two infinite parallel plates spaced 10 mm apart are inclined by 30 deg. Water flows between the plates with the volume rate of $5\ m^2/s$ (per unit width of the plate). Using Navier-Stokes equation determine:
(a) the pressure drop per unit length
(b) shear stress acting on the lower plane
(c) velocity along the centerline of the channel



Solution:
Let's chose a coordinate system with an x-axis parallel to the channel. Than the Navier-Stokes equation will read:

$$0 = -\frac{\partial p}{\partial x} + \rho g \sin(\alpha) + \mu \frac{\partial^2 u}{\partial y^2}$$

$$0 = -\frac{\partial p}{\partial y} + \rho g \cos(\alpha)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Solving the first equation: $u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x} - \rho g \sin(\alpha)\right)\left(y^2 - h^2\right)$

To find the volume rate we integrate over the height:

$$Q = \frac{2h^3}{3\mu}\left(\frac{\partial p}{\partial x} - \rho g \sin(\alpha)\right)$$

$$\frac{\partial p}{\partial x} = \frac{3\mu Q}{2h^3} + \rho g \sin(\alpha)$$

$$\frac{\partial p}{\partial x} = 5\,\text{kPa/m}$$

(b). Shear stress can be found as:

$$\tau = \mu \frac{\partial u}{\partial y} = \left(\frac{\partial p}{\partial x} - \rho g \sin(\alpha)\right) y$$

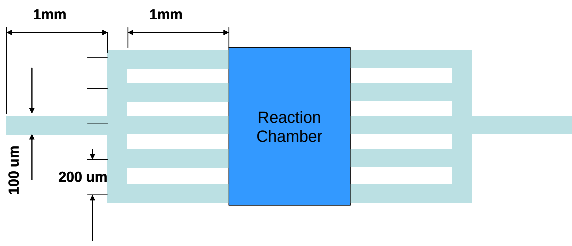
$$\tau(-h) = -3\,\text{Pa}$$

(c).

$$u(0) = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x} - \rho g \sin(\alpha)\right) h^2$$

$$u(0) = 75\,\text{m/s}$$

3. To achieve a more uniform flow distribution across the microfluidic reaction chamber you have introduced an extra circuit shown in the figure. The height of all channels – 50um, the width - 100 um (see the Figure). Estimate the extra flow resistance introduced by the circuit (i.e. neglecting flow resistance of the reaction chamber). Assume that fully developed laminar flow in all channels. What extra pressure drop will it cause at the flow rate of 20ul/min.



Solution: The circuit is made with the channels of the same cross-section, so the flow resistance per unit length is the same. Using our circuit theory we can calculate the total resistance of the device:

R_total=R_half+R_chamber+R_half

R_extra=2*R_half

$$R\_half := R1 + \left[R1^{-1} + 2\left[R2 + \left[R1^{-1} + (R2 + R1)^{-1}\right]^{-1}\right]^{-1}\right]^{-1}$$

where R1 is the resistance of 1mm long horizontal channel and R2 is the resistance of short 200um vertical channels.
The resistance per unit length for a rectangular channel can be calculated using the hydraulic diameter approximation* or the exact solution:

$$R\_br(L) := \frac{12 \cdot \mu \cdot L}{\left[1 - 0.63 \cdot \left(\frac{h}{w}\right)\right] \cdot h^3 \cdot w}$$

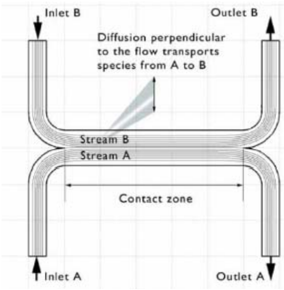
Pressure drop can be further calculated as $\Delta P\_extra := 2R\_half \cdot Q$
where Q is the volume flow rate.

Answer: 1.3 kPa.

***)**Comment: The same result is possible to obtain with the friction factor and hydraulic diameter approach.. Hydraulic resistance is defined as $R = \Delta p / Q$, where Q is the volumetric rate and ΔP is the pressure drop. ΔP can be expressed as in the equation MYO 8.33, p.431

$\Delta p = f \frac{l}{D_h} \frac{\rho V^2}{2}$, where $D_h = 4A/P$ and the product of f*Re for pipes of various cross-section can be found in the table MYO 8.3, p.448. Of course you will not get the exact equation for the resistance of a rectangular channel, but the result should be similar.

You want to distribute 30 nm diameter gold nanoparticles equally between two streams A and B using an H-cell (channel height - 50um, channel width - 50um, contact zone length 20 mm). Estimate the maximal volume flow rate you can use. Suggest how you will modify the device to improve the throughput.



Solution: Nanoparticles redistribution between the flows in the H-cell is governed by particles **diffusion across the channels** (i.e. the **width** of the channel is the dimension of importance for diffusion process). On the other hand the **time available** for diffusion is defined by the **length** of the channel and the **flow rate**. For the purpose of estimate we can use planar (2D) diffusion into semi-infinite space.

Characteristic time for diffusion across the channel: $\tau = \dfrac{w^2}{4D}$

Characteristic time for travel along the channel: $\tau = L/v = L \cdot w^2 / Q$, where v is a flow velocity and Q is a volume flow rate.

Equating those two times and using Einstein-Stokes equation for the diffusion constant:

$D = \dfrac{kT}{6\pi \cdot r \cdot \mu}$, where r is the particle radius and μ is viscosity of the solution (here, water).

Thus, $Q_{\max} = 4DL$

Answer: ~1 nl/s. The device throughput can be increased by e.g. increasing the contact length or temperature**.**